Review: Implicit Differentiation - 10/31/16

1 Implicit Differentiation

Example 1.0.1 *Let's look at the circle* $x^2 + y^2 = 25$*. We want to find* $\frac{dy}{dx}$*. We are going to take the derivative of both sides with respect to* x *to get* $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$*. Applying our derivative rules gives us* $\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + \frac{d}{dx}y^2 = 0$*. Now what do we do with the* y^2*? I could use the power rule, but that would be taking the derivative with respect to* y*, namely looking at* $\frac{d}{dy}y^2$*. But I want* $\frac{d}{dx}$*. How do we fix this? I can use the Chain Rule! I can rewrite* $\frac{d}{dx}y^2 = \frac{d}{dy}y^2 \cdot \frac{dy}{dx}$*. When I do this, I get* $2y \cdot \frac{dy}{dx}$*. That means in total, I have* $2x + 2y\frac{dy}{dx} = 0$*. I'm trying to figure out what* $\frac{dy}{dx}$ *is, so I just solve for it:* $\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$*.*

Example 1.0.2 *If* $x^3 + y^3 = 6xy$*, find* $\frac{dy}{dx}$*. Let's start by taking the derivative of both sides:* $\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}6xy$*. Let's try thinking about this in a different way: every time I take a derivative of something, I am going to mark it by writing down* $\frac{d\_}{dx}$ *where I fill in the blank with what letter I used. So here, first I took the derivative of* x^3*, so that is* $3x^2$*. I follow it by writing* $\frac{dx}{dx}$ *because I used the power rule on* x*. Next I have* $\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$ *since I used the power rule on* y*. Now for the* $6xy$*, I'm going to need the product rule, so* $f(x) = 6x$ *and* $g(y) = y$*, so* $\frac{d}{dx}6x = 6\frac{dx}{dx}$ *and* $\frac{d}{dx}y = 1\frac{dy}{dx}$*. Then using the product rule, I have* $\frac{d}{dx}6xy = 6\frac{dx}{dx}(y) + \frac{dy}{dx}(6x)$*. Note that* $\frac{dx}{dx} = 1$ *since the derivative of* x *is 1. So putting this all together, we have* $3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$*. We can simplify this a little by dividing both sides by 3 to get* $x^2 + y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$*. Now we need to solve for* $\frac{dy}{dx}$*. We can do this by getting all of the* $\frac{dy}{dx}$ *on one side and everything else on the other:* $y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - x^2$*, so* $(y^2 - 2x)\frac{dy}{dx} = 2y - x^2$*. Dividing through, we get*

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}.$$

Example 1.0.3 *Let* $4x^2 + 9y^2 = 36$*. Find the equation for the tangent line at* $(0, 2)$*. Since we already have a point, we need to find the slope of the tangent line (aka the derivative). Taking the derivative of both sides gives* $8x\frac{dx}{dx} + 18y\frac{dy}{dx} = 0$*, so* $\frac{dy}{dx} = \frac{-8x}{18y}$*. If we evaluate this at the point* $(0, 2)$*, we get that the slope is 0. Thus our line is* $y - 2 = 0(x - 0)$*, so* $y = 2$*.*

Example 1.0.4 *Let* $y = \sin(3x + 4y)$*. Find* $\frac{dy}{dx}$*. Taking the derivatives of both sides gives* $\frac{d}{dx}y = \frac{d}{dx}\sin(3x + 4y)$*, so* $1\frac{dy}{dx} = \cos(3x + 4y) \cdot (\frac{d}{dx}(3x + 4y)) = \cos(3x + 4y) \cdot (3\frac{dx}{dx} + 4\frac{dy}{dx})$*. Now we need to solve for* $\frac{dy}{dx}$*. We have* $\frac{dy}{dx} - 4\cos(3x + 4y)\frac{dy}{dx} = 3\cos(3x + 4y)$*, so*

$$\frac{dy}{dx} = \frac{3\cos(3x + 4y)}{1 - 4\cos(3x + 4y)}.$$

2 Inverse Trig Derivatives

Example 2.0.5 *Find* $\frac{dy}{dx}$ *for* $y = \arcsin(x)$*. As long as* $-\pi/2 \leq y \leq \pi/2$*, we can rewrite this as* $\sin(y) = x$*. Now we can take the derivative using implicit differentiation:* $\cos(y) \cdot \frac{dy}{dx} = 1\frac{dx}{dx}$*, so*

$\frac{dy}{dx} = \frac{1}{\cos(y)}$. *Since* $\cos^2(y)+\sin^2(y) = 1$, *then* $\cos(y) = \pm\sqrt{1-\sin^2(y)}$. *But* $\cos(y)$ *is positive between* $[-\pi/2, \pi/2]$, *so we have* $\cos(y) = \sqrt{1-\sin^2(y)}$. *But* $\sin(y) = x$, *so we have* $\cos(y) = \sqrt{1-x^2}$. *Thus*

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1.$$

Example 2.0.6 *What is* $\frac{d}{dx}\arcsin(x^2-3)$? *We can use the chain rule: let* $f(u) = \arcsin(u)$ *and* $g(x) = x^2-3$, *so* $f'(u) = \frac{1}{\sqrt{1-x^2}}$ *and* $g'(x) = 2x$. *Then* $\frac{d}{dx}\arcsin(x^2-3) = \frac{1}{\sqrt{1-(x^2-3)^2}} \cdot 2x$.

Example 2.0.7 *Find* $\frac{dy}{dx}$ *for* $y = \arccos(x)$. *As long as* $0 \le y \le \pi$, *we can rewrite this as* $\cos(y) = x$. *Now we can take the derivative using implicit differentiation:* $-\sin(y)\frac{dy}{dx} = 1\frac{dx}{dx}$, *so* $\frac{dy}{dx} = -\frac{1}{\sin(y)}$. *Since* $\cos^2(y)+\sin^2(y) = 1$, *then* $\sin(y) = \pm\sqrt{1-\cos^2(y)}$. *But we have a domain of* $0 \le y \le \pi$, *and* sin *is always positive there, so we have* $\sin(y) = \sqrt{1-\cos^2(y)}$. *Since* $\cos(y) = x$, *we can rewrite this as* $\sin(y) = \sqrt{1-x^2}$. *Thus*

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1.$$

Example 2.0.8 *What is* $\frac{d}{dx}\frac{\arccos(x)}{x}$? *We can use the quotient rule: let* $f(x) = \arccos(x)$ *and* $g(x) = x$, *so* $f'(x) = -\frac{1}{\sqrt{1-x^2}}$ *and* $g'(x) = 1$. *Then* $\frac{d}{dx}\frac{\arccos(x)}{x} = \frac{-\frac{x}{\sqrt{1-x^2}} - \arccos(x)}{x^2}$.

Example 2.0.9 *Find* $\frac{dy}{dx}$ *for* $y = \arctan(x)$. *As long as* $-\pi/2 < y < \pi/2$, *we can rewrite this as* $\tan(y) = x$. *Then using implicit differentiation, we can take the derivative:* $\sec^2(y)\frac{dy}{dx} = 1\frac{dx}{dx}$, *so* $\frac{dy}{dx} = \frac{1}{\sec^2(y)}$. *Recall that* $1+\tan^2(y) = \sec^2(y)$, *and since* $\tan(y) = x$, *then* $\sec^2(y) = 1+x^2$. *Thus*

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Example 2.0.10 *Find* $\frac{d}{dx}e^{\arctan(x)}$. *We can use the chain rule: let* $f(u) = e^u$ *and* $g(x) = \arctan(x)$, *then* $f'(u) = e^u$ *and* $g'(x) = \frac{1}{1+x^2}$. *Then* $\frac{d}{dx}e^{\arctan(x)} = e^{\arctan(x)} \cdot \frac{1}{1+x^2}$.

Practice Problems

1. If $x^3 + y^3 = 4$, find $\frac{dy}{dx}$.
2. If $x = \sqrt{x^2+y^2}$, find $\frac{dy}{dx}$.
3. If $e^{xy} = e^{4x} + e^{5y}$, find $\frac{dy}{dx}$.
4. Find the equation of the tangent line of $x^2y + y^4 = 4 + 2x$ at the point $(-1, 1)$.

Solutions

1. Taking the derivative of both sides gives $\frac{d}{dx}x^3 + \frac{d}{x}y^3 = \frac{d}{dx}4$, so $3x^2\frac{dx}{dx} + 3y^2\frac{dy}{dx} = 0$. Now we can solve for $\frac{dy}{dx}$, we get $\frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}$.

2. We need to use the chain rule to take the derivative of $\sqrt{x^2+y^2}$. This gives us $\frac{1}{2\sqrt{x^2+y^2}} \cdot \left(2x\frac{dx}{dx} + 2y\frac{dy}{dx}\right)$. Thus we have $1\frac{dx}{dx} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \left(2x\frac{dx}{dx} + 2y\frac{dy}{dx}\right) = \frac{x+y\frac{dy}{dx}}{\sqrt{x^2+y^2}}$. Solving for $\frac{dy}{dx}$ gives $\sqrt{x^2+y^2} = x + y\frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}-x}{y}$.

3. If we use the chain rule for all of these, we get $e^{xy} \cdot (\frac{dx}{dx}y + \frac{dy}{dx}x) = e^{4x} \cdot 4\frac{dx}{dx} + e^{5y} \cdot 5\frac{dy}{dx}$. Getting all of the $\frac{dy}{dx}$ on one side gives us $xe^{xy}\frac{dy}{dx} - 5e^{5y}\frac{dy}{dx} = 4e^{4x} - ye^{xy}$, so solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{4e^{4x} - ye^{xy}}{xe^{xy} - 5e^{5y}}.$$

4. Since we already have a point, we just need the slope to find the equation of the line. We can get this by finding the derivative using implicit differentiation: $(2x\frac{dx}{dx}y + \frac{dy}{dx}x^2) + 4y^3\frac{dy}{dx} = 2\frac{dx}{dx}$. Bringing all the $\frac{dy}{dx}$ to one side gives us $x^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 2 - 2xy$, so $\frac{dy}{dx} = \frac{2-2xy}{x^2+4y^3}$. Now to find the slope of the tangent line at the point $(-1, 1)$, we just need to plug those values into the derivative to get $\frac{2-2(-1)(1)}{(-1)^2+4(1)^3} = \frac{4}{5}$. Then using point slope form, we get $y - 1 = \frac{4}{5}(x - (-1))$, so $y = \frac{4}{5}x + \frac{9}{5}$.